DC STATES OF PLASMA DIODES

National Aeronautics and Space Administration
Grant NsG- 29963
Project #0833

Project Leader: H. Derfler
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Quarterly Status Report Q.R.R. #9
1 April - July 1, 1964

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OBJECTIVES

The purpose of this project is the study of stationary potential distributions between parallel plane electron and ion emitting surfaces.

SUMMARY

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In view of the importance of charge exchange collisions in certain phenomena related to low pressure thermionic energy converters, the mobility of positive ions subject to such collisions has been reviewed. An expression due to Sheldon for the charge exchange mobility at low electric fields is considered basically sound. However an equation for the drift velocity at high electric fields given by Sena has to be rejected because of his underlying assumption that the ions created start from rest. By properly balancing the momentum transfer in a charge exchange collision we have arrived at an integro differential equation for the ion distribution function. Work is in progress to solve this equation in order to find an expression for the charge exchange mobility at high electric fields.

Author

A. INTRODUCTION

In an investigation of the open circuit voltage of a cesium plasma diode a theoretical expression for the mean velocity of ions subject to charge exchange collisions was used. This expression

$$\langle v_+ \rangle = \sqrt{\frac{\pi e E}{4 n_o m_+ Q}} \qquad (1)$$

was obtained by Sena[1] subject to the following conditions:

i) The charge exchange cross section Q_x is independent of the relative kinetic energy $\epsilon = \frac{M}{2} (\vec{v} - \vec{V})^2$ of the approaching particles.

ii) The temperature of the neutral particles is negligible $kT_o = 0$

iii) Each ion looses its kinetic energy completely upon impact with a neutral atom and so after charge exchange starts movingagain from rest $\vec{v} = 0$ in the accelerating electric field $\vec{E}$.

iv) The quantum mechanical exchange energy is neglected.

It has been pointed out by Sheldon[2] that the charge exchange cross section Q_x is dependent on the energy ϵ of approach due to the large atomic polarizability $\alpha = 52.5 \cdot 10^{-24}$ cm^3 [3] of the

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1) L.A. Sena, J. Exp. Theor. Phys. (USSR) 16, 734, 1946.

2) J.W. Sheldon, J. Appl. Phys. 34, 444 (1963).

3) A. Salop, E. Pollack and B. Bederson, Phys. Rev. 124, 1431 (1961).

neutral cesium atom. He uses an approximation of the type

$$Q_x = (A - B\ \ell n\epsilon)^2 + \frac{\alpha}{\epsilon}\left(\frac{e\pi}{2}\right)^2 (A - B\ \ell n\epsilon)^{-2} \tag{2}$$

Assuming that all collisions are head-on he then calculates an average cross section for momentum transfer

$$\langle Q\rangle = \frac{1}{(kT)^3}\int_0^\infty \epsilon^2 Q_x e^{-\epsilon/kT}\, d\epsilon \tag{3}$$

and obtains a drift velocity

$$\langle \vec{v}_+\rangle = \frac{3}{8}\sqrt{\frac{\pi m}{kT}}\ \frac{e\vec{E}}{mn_o\langle Q\rangle} \tag{4}$$

We note that in deriving eq. (4) it is inherently assumed that the drift velocity is much smaller than the mean directed thermal velocity that is $\langle v_+\rangle \ll (8kT/\pi m)^{1/2}$ or by eq. (4)

$$E_c \cong \frac{16.2^{1/2}}{3\pi}\ \frac{kT}{e\lambda_+} \gg E \tag{5}$$

where $\lambda_+ = 1/n_o\ \langle Q\rangle$ is the ion mean free path. There is no real discrepancy between eq. (1) and eq. (4) because Sena's expression is supposedly valid for large electric fields while Sheldon's expression applies for low electric fields. It is no wonder that the charge exchange cross sections obtained from mobility measurements vary widely depending on whether the condition $E_c << E$ or $E_c >> E$ has been maintained experimentally and whether the proper eq. (1) or eq. (4) respectively has been used to compute $\langle Q \rangle$.

When applying the concept of charge exchange mobility to the low pressure Cesium diode conditon, eq. (5) is almost always violated and so Sena's formula (1) must be used[4]. There is, however, considerable doubt that Sena's assumption iii), which amounts to counting only head-on collisions, is still correct in the high field limit. This was the motivation for an investigation which shows that Sena's formula eq. (1) is indeed on very weak ground. Work in progress to correct this situation is outlined below.

B. THE KINETIC EQUATIONS FOR CHARGE EXCHANGE COLLISIONS

We attempt to calculate the charge exchange mobility of positive ions in their own gas by dropping all of Sena's restrictions but ii) and iv). Neglect of the quantum mechanical exchange energy allows us to treat a charge exchange collision like an elastic collision with the only provision being that the identity of the particles have changed "after" collision. To illustrate exactly what is meant under "change of identity" we show in Figs. 1a and 1b a direct and an inverse encounter between an ion and a neutral atom which does not lead to charge exchange. In Figs. 2a and 2b we show the same

4) H. Derfler, QRR No. 7, Stanford Electronics Laboratories, Electron Devices Laboratory, Stanford University.

encounter with charge exchange taking place. We shall use the conventional symbols $\vec{v}$, $\vec{V}$ and $\vec{v}'$, $\vec{V}'$ for corresponding velocities of particles before and after collision, respectively. These velocities will not be labeled to indicate their charge as it is sufficient to identify their electrical nature by a subscript "o" for neutrals and "+" for ions to the distribution function in which they appear as argument. With this convention the Boltzmann equation for the ion distribution function is written in the form

$$\frac{d}{dt} f_+(\vec{v}) = \iint \sigma_{++}(\chi)|\vec{v} - \vec{V}|\left\{f_+(\vec{v}')f_+(\vec{V}')-f_+(\vec{v})f_+(\vec{V})\right\} d\Omega d\vec{V}$$

$$+ \iint \sigma_{+o}(\chi) \, |\vec{v} - \vec{V}|\left\{f_+(\vec{v}')F_o(\vec{V}')-f_+(\vec{v})F_o(\vec{V})\right\} d\Omega d\vec{V}$$

$$+ \iint \sigma_x(\chi)|\vec{v} - \vec{V}|\left\{f_+(\vec{V}')F_o(\vec{v}') - f_+(\vec{v})F_o(\vec{V})\right\} d\Omega d\vec{V} \tag{6}$$

where

$$d\Omega = \sin \chi d\chi d\phi \tag{7}$$

is the solid angle element of the scattering sphere and

$$\sigma_{ik}(\chi) = \frac{b_{ik}}{2 \sin \chi} \left(\frac{\partial b_{ik}}{\partial \chi}\right) |\vec{v} - \vec{V}| \tag{8}$$

is the differential scattering cross section as defined by the impact parameter b_{ik}[5]. Note that the first integral balances

5) S. Chapman and T.G. Cowling, "The Mathematical Theory of Non-Uniform Gases", Cambridge University Press, 1952, Chapter 10.

ion-ion collisions, the second elastic neutral atom-ion collisions and the last charge-exchange collisons between neutral atoms and ions. Note the important fact that the velocities $\vec{v}'$ and $\vec{V}'$ of the inverse collision in the 3rd integral are interchanged as compared with the 2nd integral. This is in fact the mathematical expression for the "change in identity" due to the charge exchange collision described above and shown in Fig. 2b. We cannot over-emphasize the fact that <u>charge exchange appears only in the balance of inverse collisions!</u>

We shall now take care of Sena's condition ii) assuming that the bulk of the neutral atoms is cold. Thus restricting ourselves to the case of large applied electric field (compare eq. 5) we use the Ansatz

$$F_o(\vec{V}) = n_o\delta(\vec{V}) + f_o(\vec{V}) \qquad (9)$$

where $\delta(\vec{V})$ is a Dirac delta function representing the bulk of the neutral atoms and $f_o(\vec{V})$ is a small contribution due to "fast" neutral atoms. We also assume that the gas is only weakly ionized such that $f_+(\vec{v})$ is small and products like $f_o(\vec{V})f_+(\vec{v})$ can be neglected. With this provision also the ion-ion interaction can be neglected $\sigma_{++} = 0$ and to focus our attention to the charge exchange mechanism we also discard the elastic collisions between neutral atoms and ions $\sigma_{+o} = 0$. With these assumptions and eq. (6) we linearize the Boltzmann equation (6) and find

$$\frac{d}{dt}f_+(\vec{v})+n_oQ_x|\vec{v}|f(\vec{v})=n_o\iint\sigma_x(\gamma)\ |\vec{v}-\vec{V}|f_+(\vec{V}')\delta_o(\vec{v}')\ d\Omega d\vec{V} \qquad (10)$$

where

$$Q_x = \int \sigma_x(\gamma)\ d\Omega \qquad (11)$$

is the total collision cross section for charge transfer as given for example by eq. (2). This then is the kinetic equation describing the charge exchange effect at large electric fields.

C. TRANSFORMATION OF THE COLLISION INTEGRAL

To make efficient use of the Dirac δ-function in the integrand of eq. (10) we would like to integrate with respect to $\vec{v}'$ instead of $\vec{V}$. The Jacobian involved in this transformation has to be evaluated at constant $\vec{v}$, γ, $\emptyset$ and was found to be*

$$\left\|\frac{\partial\vec{V}}{\partial\vec{v}'}\right\|_{\vec{v},\ \gamma,\ \emptyset} = \left|\frac{\vec{v}-\vec{V}}{\vec{v}-\vec{v}'}\right|^3 \frac{\sin\vartheta_r}{\sin\vartheta\cos(\varphi_r-\varphi)} \qquad (12)$$

where ϑ_r, φ_r and ϑ, φ are polar angles of $\vec{v}-\vec{V}$ and $\vec{v}-\vec{v}'$ with respect to a direction $\vec{e}$ fixed in space which we shall later identify with the direction of the electric field vector. Considerable time has gone into checking and double checking this Jacobian because Allis[6] has obtained a Jacobian under

6) W.P. Allis, "Motion of Ions and Electrons", Article in "Encyclopedia of Physics" Springer Verlag, Berlin, 1956, p. 409.

* See Appendix A

similar conditions but missing the trigonometrical factor given in eq. (12). The consequences of his error will be dealt with in a separate publication.

Introducing eq. (11) into eq. (9) we find

$$\frac{d}{dt} f_+(\vec{v}) + n_o Q_x |\vec{v}| f(\vec{v}) = n_o \iint \sigma_x(\nu) |\vec{V}'| f_+(\vec{V}') \left\| \frac{\partial \vec{V}}{\partial \vec{v}'} \right\|_{\vec{v}, \nu, \emptyset; \vec{v}'=0} d\Omega \tag{13}$$

We note that due to the restriction $\vec{v}' = 0$ arising from the Dirac delta function in integral eq. (10) the velocities $\vec{V}'$ and $\vec{V}$, given $\vec{v}$, are no longer independent but functions of ν and $\emptyset$. From the laws of Conservation of Momentum and Energy in an elastic collision, subject to the condition $\vec{v}' = 0$, we get a relation between the velocity vectors which is shown in Fig. 3. From this graph we find immediately that

$$\sin \frac{\nu}{2} = \left| \frac{\vec{v}}{\vec{V}'} \right| \qquad \frac{d\nu}{d|\vec{V}'|} = \frac{2}{|V'|} \frac{\sin^2 \nu/2}{\cos \nu/2} d|\vec{V}'| \tag{14}$$

and thus we can choose to integrate eq. (13) with respect to $|\vec{V}'|$ instead of ν. We also would like to change the variable $\emptyset$ to ϑ'. The variation of $\emptyset$ with ϑ' when $\vec{v}$, $\vec{v}' = 0$, $|\vec{V}'|$ (that is ν) are held fixed is obtained from the orientation of the plane Fig. 3 with respect to an axis $\vec{e}$ fixed in space by considerations of spherical trigonometry. The result is given by

$$\left(\frac{d\phi}{d\vartheta'}\right)_{\gamma,\vartheta} = \frac{\sin\vartheta\cos(\varphi_r-\varphi)}{\sin\vartheta_r\,\sin\beta\,\cos\gamma/2} \tag{15}$$

where β is the angle between the vectors $\vec{e} \times \vec{V}'$ and $\vec{v} \times \vec{V}'$ respectively.*

Collecting equations (7), (12), (14) and (15) we find

$$d\Omega \left\| \frac{\partial\vec{V}}{\partial\vec{v}'} \right\|_{\vec{v},\gamma,\phi;\vec{v}'=0} = \frac{4\,d|\vec{V}'|\,d\vartheta'}{|\vec{v}|\,\cos\gamma/2\,\sin\beta} \tag{16}$$

It is necessary to express β in terms of γ, ϑ and ϑ' which again is done by means of spherical trigonometry.*

$$\sin\beta = \frac{1}{\sin\vartheta'\cos\gamma/2}\sqrt{\cos^2\frac{\gamma}{2} - \cos^2\vartheta - \cos^2\vartheta' + 2\sin\vartheta\sin\vartheta'\sin\frac{\gamma}{2}} \tag{17}$$

By introducing equations (14), (16) and (17) into equation (13) we obtain

$$\frac{d}{d\tau} f_+(\vec{v}) + n_o Q_x |\vec{v}|\, f_+(\vec{v}) =$$

$$= \frac{4n_o}{|\vec{v}|}\iint \frac{\sigma_x(\gamma,|\vec{V}'|)f_+(\vec{V}')\,\sin\vartheta'd\vartheta'|\vec{V}'|d|\vec{V}'|}{\sqrt{1-\left|\frac{\vec{v}}{\vec{V}'}\right|^2 + 2\left|\frac{\vec{v}}{\vec{V}'}\right|\sin\vartheta\sin\vartheta' - \cos^2\vartheta - \cos^2\vartheta'}} \tag{18}$$

which is identical with integral equation (10).

*See Appendix B

D. THE CHARACTERISTICS OF THE DIFFERENTIAL OPERATOR EQ. (18)

In a homogeneous neutral plasma and under stationary conditions we have simply

$$\frac{d}{dt} f_+(\vec{v}) = \frac{e}{m} \vec{\mathcal{E}} \cdot \frac{\partial}{\partial \vec{v}} f_+(\vec{v}) = \frac{e}{m} \mathcal{E} \frac{\partial}{\partial v_{\parallel}} f_+(\vec{v}) \tag{19}$$

It is convenient to introduce in eq. (19) spherical coordinates in velocity space so that eq. (18) becomes under said conditions:

$$L\left\{f_+(v)\right\} = \frac{4\pi n_o}{e\mathcal{E}} \iint \frac{\sigma_x(\chi, |\vec{v}'|) f_+(\vec{v}') \sin\vartheta' d\vartheta' |\vec{v}'| d|\vec{v}'|}{\sqrt{1 - \left|\frac{\vec{v}}{\vec{v}'}\right|^2 + 2\left|\frac{\vec{v}}{\vec{v}'}\right| \sin\vartheta \sin\vartheta' - \cos^2\vartheta - \cos^2\vartheta'}} \tag{20}$$

where the linear differential operator

$$L = |\vec{v}|\cos\vartheta \frac{\partial}{\partial|\vec{v}|} + \sin^2\vartheta \frac{\partial}{\partial\cos\vartheta} + \frac{mn_oQ}{e\mathcal{E}} |\vec{v}|^2 \tag{21}$$

has been introduced. Using the variables

$$|\vec{v}| = x \quad \text{and } \cos\vartheta = y \tag{22}$$

the characteristic equations of the differential operator $L\{f\}$ can be written by means of an arbitrary parameter s:

$$\frac{dx}{ds} = xy \qquad \frac{dy}{ds} = 1 - y^2 \qquad \frac{df}{ds} = -\frac{mn_oQ}{e\mathcal{E}} x^2 f \tag{23}$$

This set of linear differential equations can be solved by elementary means giving the characteristics

$$x = x_o[\cosh(s) + y_o \sinh(s)] \tag{24}$$

$$y = \frac{y_o + \tanh(s)}{1 + y_o \tanh(s)} \tag{25}$$

$$f=f_o\, e^{-\frac{mn_oQ}{2eE} x_o^2[(1-y_o^2)s+(1+y_o^2)\sinh(s)\cdot\cosh(s)+2y_o \sinh^2(s)]} \tag{26}$$

Note that for $s = 0$ we obtain the "initial values" $x = x_o$, $y = y_o$, $f = f_o$. By prescribing the initial values in the form

$$x_o = x_o(t) \quad y = y_o(t) \quad f = f_o(t) \tag{27}$$

where t is a suitable parameter we may introduce eq. (27) into the characteristic equations (24), (25) and (26) and eliminate s,t to obtain the general solution $f(x,y)$ of the linear differential equation

$$L\{f(\vec{v})\} = 0 \tag{28}$$

One solution of eq. (28) which will be discussed further below is

$$f = f_o\, e^{-\frac{mn_oQ}{2eE} x^2\{y + C(1-y^2) + (1-y^2) \ln\sqrt{\frac{1+y}{1-y}}\}} \tag{29}$$

where C is an arbitrary constant.

Actually we are not very much interested in solutions of the homogeneous equation (28) but we hope that by means of the characteristics eqs. (24), (25) and (26) we can transform the differential operator (21) into an integral operator. In this way we try to transform the integro differential equation (20) into a pure integral equation which we hope to solve, if necessary, by mumerical methods.

3. DISCUSSION

If we take Sena's condition iii) seriously each ion created will start from rest and move parallel to the electric field until it collides again head-on with a neutral atom. This amounts to replacing the "generation term", that is the integral over the inverse collisions in eq. (20) by a Dirac-δ function $\delta(\vec{v})$ and allowing only for positive velocities parallel to the electric field. In this case we have from eqs. (18) and (19)

$$\frac{\partial}{\partial v_{\parallel}} f_{+}(v_{\parallel}) + \frac{m n_o Q_x}{eE} v_{\parallel} f_{+}(v_{\parallel}) = 0, \quad v_{\parallel} > 0 \qquad (30)$$

The solution of this equation is

$$f_{+}(v_{\parallel}) = f_o \, e^{-\frac{m n_o Q_x}{2eE} v_{\parallel}^2}, \quad v_{\parallel} > 0 \qquad (31)$$

from which the drift velocity of the positive ions is found to be

$$\langle v_{\parallel} \rangle = \frac{\int_0^\infty v_{\parallel} \, e^{\frac{-mn_oQ}{2eE} v_{\parallel}^2} dv_{\parallel}}{\int_0^\infty e^{\frac{-mn_oQ}{2eE} v_{\parallel}^2} dv_{\parallel}} = \sqrt{\frac{2eE}{\pi mn_oQ_x}} \tag{32}$$

Comparing this with Sena's equation (1) we find agreement apart from a numerical factor of order unity. The disagreement of the numerical factor must not be taken too seriously because Sena's formula was derived by little more than dimensional analysis. A serious objection to both equations (1) and (32) arises by considering the effect of transverse velocities. Assuming with Sena that the "generation term" can be lumped into a Dirac δ-function $\delta(\vec{v})$ eq. (28) is valid everywhere except for $\vec{v} = 0$ and the same holds for the solution eq. (29). By substituting eq. (22) into eq. (29) and taking $C = 0$ we find that

$$f = f_o \, e^{\frac{-mn_oQ}{2eE} |\vec{v}|^2 \left\{ \cos\vartheta + \sin^2\vartheta \, \ln\left(\cot g \frac{\vartheta}{2}\right) \right\}} \tag{33}$$

which for $\vartheta = 0$ agrees exactly with eq. (31). It is not obvious why we should average eq. (33) only over positive velocities as it was done in eq. (32) to obtain the drift velocity. If instead we try to average (33) over all velocities we are in serious trouble because the distribution function eq. (33) blows up for $|\vartheta| > \pi/2$ as shown in Fig. 4. From this figure we see that we get even more ions moving against the electric field than moving in its direction. This does not make sense physically and to find the correct distribution function we must solve the full integro differential equation (20) as outlined in Section D. Work along this line is now in progress.

Appendix A: The Transformation $\vec{V} \rightarrow \vec{v}'$

We wish to calculate the Jacobian eq. (12) given in the text.

Let the velocities of two particles with mass m_1 and m_2 before and after collision be $\vec{v}$, $\vec{V}$ and $\vec{v}'$, $\vec{V}'$ respectively. Since momentum and energy are conserved in an elastic collision these velocities can be written

$$\vec{v} = \vec{v}_g + M_2\vec{v}_r \qquad \vec{v}' = \vec{v}_g + M_2\vec{v}_r' \qquad M_2 = \frac{m_2}{m_1 + m_2} \tag{A1}$$

$$\vec{V} = \vec{v}_g - M_1\vec{v}_r \qquad \vec{V}' = \vec{v}_g - M_1\vec{v}_r' \qquad M_1 = \frac{m_1}{m_1 + m_2} \tag{A2}$$

where

$$\vec{v}_g = M_1\vec{v} + M_2\vec{V} = M_1\vec{v}' + M_2\vec{V}' \tag{A3}$$

is the velocity of the center of gravity and

$$\vec{v}_r = \vec{v} - \vec{V}, \qquad \vec{v}_r' = \vec{v} - \vec{V}', \qquad |\vec{v}_r| = |\vec{v}_r'| \tag{A4}$$

are the relative velocities before and after impact respectively.

It follows immediately from (A1) and (A2) that the <u>end points</u> of the vectors $\vec{v}$, $\vec{V}$, $\vec{v}'$ $\vec{V}'$ and $\vec{v}_g$ in velocity space are all in one and the same plane the normal of which is in the direction of

$$\vec{p} = \frac{\vec{v}_r \times \vec{v}_r'}{|\vec{v}_r \times \vec{v}_r'|} \tag{A5}$$

That is all that can be deduced from the conservation laws. Given $\vec{v}$ and $\vec{V}$ relations (A3) and (A4) constitute only four equations for the six components of $\vec{v}'$ and $\vec{V}'$ and so we need two more parameters to specify the collision completely. As such it is conventional[7] to specify the scattering angle ν in the center of gravity system

$$\cos \nu = \frac{\vec{v}_r \cdot \vec{v}_r{}'}{|\vec{v}_r|^2} \tag{A6}$$

and the orientation of the plane of the relative motion. If

$$\vec{q} = \frac{\vec{e} \times \vec{v}_r}{|\vec{e} \times \vec{v}_r|} \tag{A7}$$

is the normal of a plane through the relative velocity $\vec{v}_r$ and a **direction $\vec{e}$ fixed in space** the angle φ,

$$\cos \varphi = \vec{p} \cdot \vec{q} \tag{A8}$$

is used to measure the direction of $\vec{p}$. Given $\vec{e}$, this angle φ is determined from the initial conditions, e.g. from the relative position $\vec{x} - \vec{X}$ and velocity $\vec{v} - \vec{V}$ of the two particles before encounter ($t = -\infty$). The scattering angle ν is found only if in addition the force law of interaction is specified and after Newton's equations of the relative motion have been integrated[7]. By means of the angle ν a velocity diagram

7) S. Chapman and T.G. Cowling, "The Mathematical Theory of Non-Uniform Gases", Cambridge University Press, 1952, Chapter 3, p. 50 and Chapter 10, p. 170.

solving eqs. (A1) and (A2) can be constructed in the plane of the relative motion as shown in Fig. 5. The orientation of this plane Fig. 5 with respect to an axis fixed in space is shown in Fig. 6.

With this recollection of the laws of elastic collisions we are now ready to perform the transformation

$$d\vec{V} = \left(\frac{\partial \vec{V}}{\partial \vec{v}'}\right)_{\vec{v},\chi,\phi} d\vec{v}' \tag{A9}$$

For constant $\vec{v}$, χ, ϕ we have

$$\frac{\partial \vec{V}}{\partial \vec{v}'} = \frac{\partial(\vec{v}-\vec{V})}{\partial(\vec{v}-\vec{v}')} = \frac{\partial(\vec{v}-\vec{V})}{\partial|\vec{v}-\vec{V}|,\ \vartheta_r,\varphi_r} \cdot \frac{\partial|\vec{v}-\vec{V}|,\ \vartheta_r,\varphi_r}{\partial|\vec{v}-\vec{v}'|,\ \vartheta,\varphi} \cdot \frac{\partial|\vec{v}-\vec{v}'|,\ \vartheta,\varphi}{\partial(\vec{v}-\vec{v}')} \tag{A10}$$

where ϑ_r, φ_r and ϑ, φ are polar angles of $\vec{v}_r = \vec{v}-\vec{V}$ and $\vec{v}-\vec{v}'$ with respect to an axis $\vec{e}$ fixed in space as shown in Fig. 6. The first and last matrix in eq. (A10) are transformations to spherical coordinates the Jacobian of which are trivial. With these we find

$$\left\|\frac{\partial \vec{V}}{\partial \vec{v}'}\right\|_{\vec{v},\chi,\phi} = \left|\frac{\vec{v}-\vec{V}}{\vec{v}-\vec{v}'}\right|^2 \frac{\sin\vartheta_r}{\sin\vartheta} \left\|\frac{\partial|\vec{v}-\vec{V}|,\vartheta_r,\varphi_r}{\partial|\vec{v}-\vec{v}'|\,\vartheta,\varphi}\right\|_{\vec{v},\chi,\phi} \tag{A11}$$

From the triangle v_c v D shown in the velocity diagram Fig. 5 we deduce that

$$|\vec{v}-\vec{V}| = 2M_2 \sin\frac{\chi}{2}\ |\vec{v}-\vec{v}'| \tag{A12}$$

and from the spherical triangle ABC in Fig. 6 we obtain

$$\cos\vartheta = \cos\vartheta_r \sin\frac{\chi}{2} + \sin\vartheta_r \cos\frac{\chi}{2}\cos\phi \tag{A13}$$

$$\sin\vartheta \sin(\varphi_r - \varphi) = \cos\frac{\chi}{2}\sin\phi \tag{A14}$$

Differentiating eqs. (A12), (A13) and (A14) at constant $\vec{v}$, χ, ϕ we find

$$\left.\begin{aligned} d|\vec{v}-\vec{V}| &= 2M_2 \sin\frac{\chi}{2}\, d|\vec{v}-\vec{v}'| \\ d\vartheta_r &= \frac{\sin\vartheta}{\sin\vartheta_r \sin\frac{\chi}{2} - \cos\vartheta_r \cos\frac{\chi}{2}\cos\phi}\, d\vartheta \\ d\varphi_r &= -\operatorname{tg}(\varphi_r-\varphi)\cot g\,\vartheta\, d\vartheta + d\varphi \end{aligned}\right\} \tag{A15}$$

The determinate of this transformation is

$$\left\| \frac{\partial\, |\vec{v}-\vec{V}|,\ \vartheta_r,\ \varphi_r}{\partial\, |\vec{v}-\vec{v}'|,\ \vartheta,\ \varphi} \right\|_{\vec{v},\chi,\phi} = \left| \frac{\vec{v}-\vec{V}}{\vec{v}-\vec{v}'} \right| \frac{\sin\vartheta}{\sin\vartheta_r \sin\frac{\chi}{2} - \cos\vartheta_r \cos\frac{\chi}{2}\cos\phi} \tag{A16}$$

It is convenient to eliminate the impact parameters χ,ϕ from this formula. Introducing $\cos\frac{\chi}{2}\cos\phi$ from eq. (A13) the denominator of eq. (A16) becomes

$$\sin \vartheta_r \sin \frac{\gamma}{2} - \cos \vartheta_r \cos \frac{\gamma}{2} \cos \emptyset = \frac{\sin \frac{\gamma}{2} - \cos \vartheta \cos \vartheta_r}{\sin \vartheta_r} \qquad (A17)$$

Also from the spherical triangle ABC in Fig. 6 we deduce

$$\sin \frac{\gamma}{2} = \cos \vartheta \cos \vartheta_r + \sin \vartheta \sin \vartheta_r \cos(\varphi_r - \varphi) \qquad (A18)$$

When substituting (A18), (A17) and (A16) into (A11) we obtain

$$\left\| \frac{\partial \vec{V}}{\partial \vec{v}'} \right\|_{\vec{v},\gamma,\emptyset} = \left| \frac{\vec{v}-\vec{V}}{\vec{v}-\vec{v}'} \right|^3 \frac{\sin \vartheta_r}{\sin \vartheta \cos(\varphi_r - \varphi)} \qquad (A19)$$

which is the Jacobian eq. (12) used in the text.

Appendix B: The Transformation $\emptyset \rightarrow \vartheta'$

We wish to calculate the variation of the azimuthal angle $\emptyset$ with the polar angle ϑ' when $\vec{v}$, $\vec{v}' = 0$ and γ are held fixed. In this case all velocities are in one and the same plane as shown in Fig. 3. The orientation of this plane with respect to an axis $\vec{e}$ fixed in space is shown in Fig. 7. Let ϑ' and φ' be the polar angles of $\vec{V}'$ then we have by the law of cosines of the spherical triangles in Fig. 7

$$\text{from ABC} \quad \cos\vartheta' = \cos\vartheta \sin\frac{\gamma}{2} + \sin\vartheta \cos\frac{\gamma}{2} \cos\gamma \qquad \text{(B1)}$$

$$\text{from ACD} \quad \cos\vartheta_r = \cos\vartheta \sin\frac{\gamma}{2} - \sin\vartheta \cos\frac{\gamma}{2} \cos\gamma \qquad \text{(B2)}$$

$$\text{adding both} \quad \cos\vartheta' + \cos\vartheta_r = 2\sin\frac{\gamma}{2}\cos\vartheta \qquad \text{(B3)}$$

$$\text{from ACD} \quad \cos\vartheta = \cos\vartheta_r \sin\frac{\gamma}{2} + \sin\vartheta_r \cos\frac{\gamma}{2} \cos\emptyset \qquad \text{(B4)}$$

differentiating eq. (B3) and (B4) at constant γ, ϑ we have

$$\sin\vartheta' \, d\vartheta' + \sin\vartheta_r d\vartheta_r = 0 \qquad \text{-(B5)}$$

$$\left(\sin\vartheta_r \sin\frac{\gamma}{2} - \cos\vartheta_r \cos\frac{\gamma}{2} \cos\emptyset\right) d\vartheta_r + \sin\vartheta_r \cos\frac{\gamma}{2} \sin\emptyset d\emptyset = 0 \qquad \text{(B6)}$$

Eliminating $d\vartheta_r$ between (B5) and (B6) using (B4) we obtain

$$\left(\frac{d\emptyset}{d\vartheta'}\right)_{\gamma,\vartheta} = \frac{\sin\vartheta' \left(\sin\frac{\gamma}{2} - \cos\vartheta_r \cos\vartheta\right)}{\sin^3\vartheta_r \cos\frac{\gamma}{2} \sin\emptyset} \qquad \text{(B7)}$$

We also wish to eliminate the angle ϕ from eq. (B7). By means of the spherical triangles shown in Fig. 7 we have

from ACD $$\sin\frac{\nu}{2} = \cos\vartheta_r\cos\vartheta + \sin\vartheta_r\sin\vartheta\cos(\varpi_r - \varpi) \tag{B8}$$

from ABD $$\sin\vartheta_r\sin\phi = \sin\vartheta'\sin\beta \tag{B9}$$

Substituting these into eq. (B7) we find

$$\left(\frac{d\phi}{d\vartheta'}\right)_{\nu,\vartheta} = \frac{\sin\vartheta'\cos(\varpi_r - \varpi)}{\sin\vartheta_r\sin\beta\cos\frac{\nu}{2}} \tag{B10}$$

which is equation (15) used in the text.

To express β in terms of ν, ϑ and ϑ' we use once more the spherical triangle ABC shown in Fig. 7:

$$\cos\vartheta = \cos\vartheta'\sin\frac{\nu}{2} + \sin\vartheta'\cos\frac{\nu}{2}\cos\beta \tag{B11}$$

from which $\sin\beta$ is calculated as used in eq. (17) of the text.

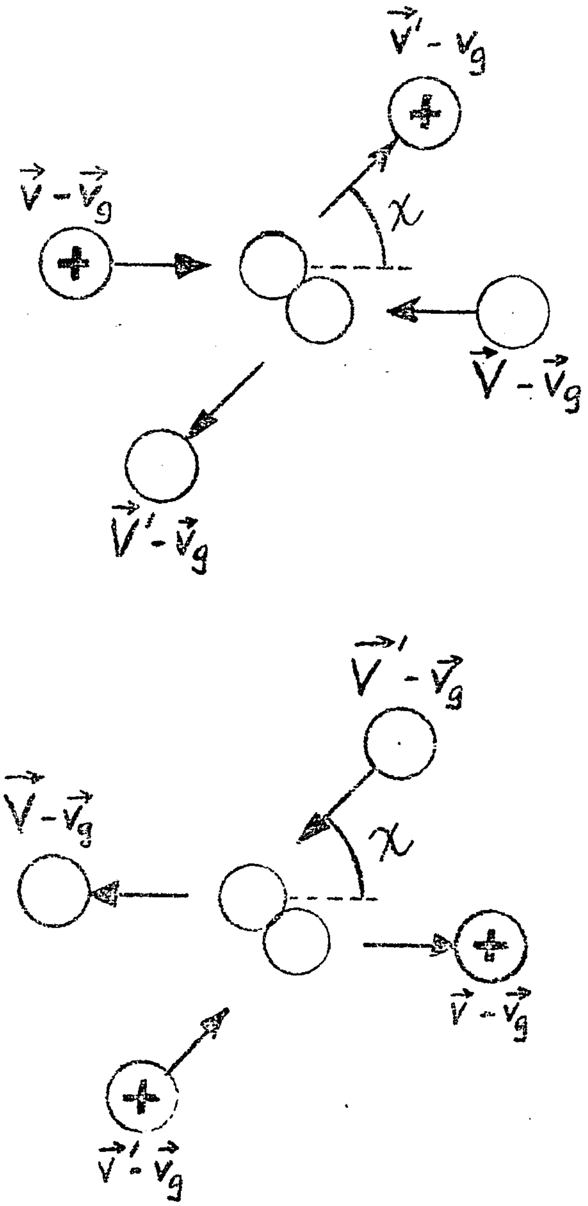

Fig. 1a and 1b
Direct encounter and below inverse encounter without charge exchange in the center of gravity system.

$\vec{v}_g$ = c.g. velocity

σ_{o+} =elastic scatterring cross section

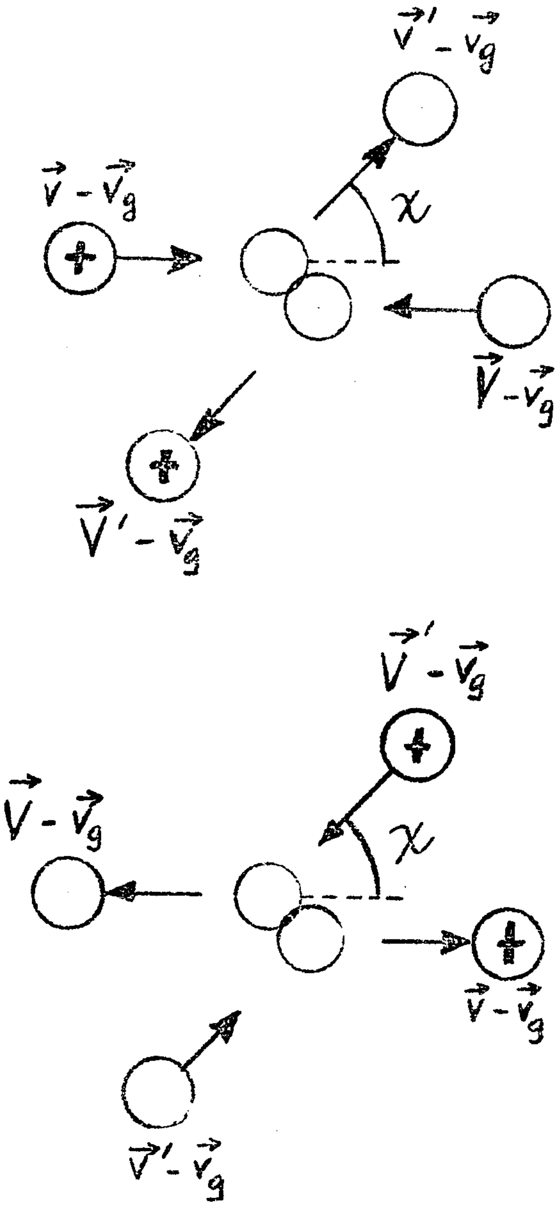

Fig. 2a and 2b
Direct encounter and below inverse encounter with charge exchange in the center of gravity system

$\vec{v}_g$ = c.g. velocity

σ_x = charge exchange cross section

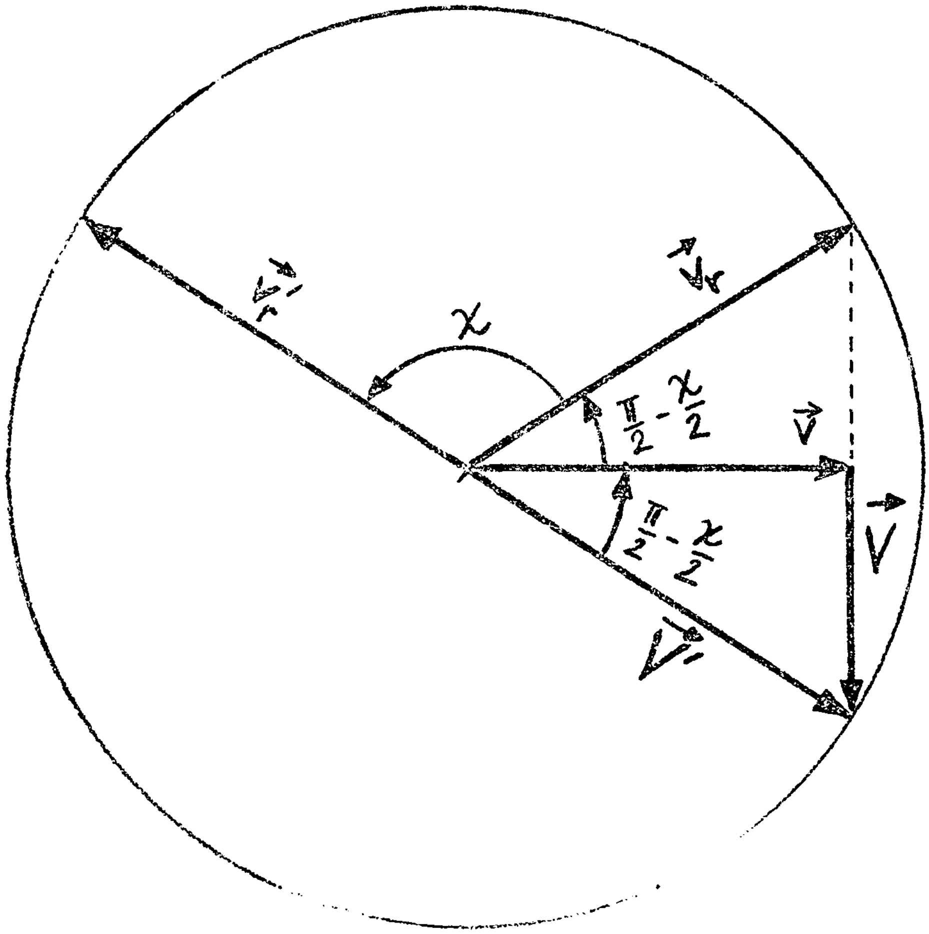

Fig. 3 Velocities in an elastic collision subject to the condition that one of the particles is at rest "after" collision $\vec{v}' = 0$.

$\vec{v}_r = \vec{v} - \vec{V}$ relative velocity before collision

$\vec{v}_r' = -\vec{V}'$ relative velocity after collision

χ= scatterring angle in the c.g. system

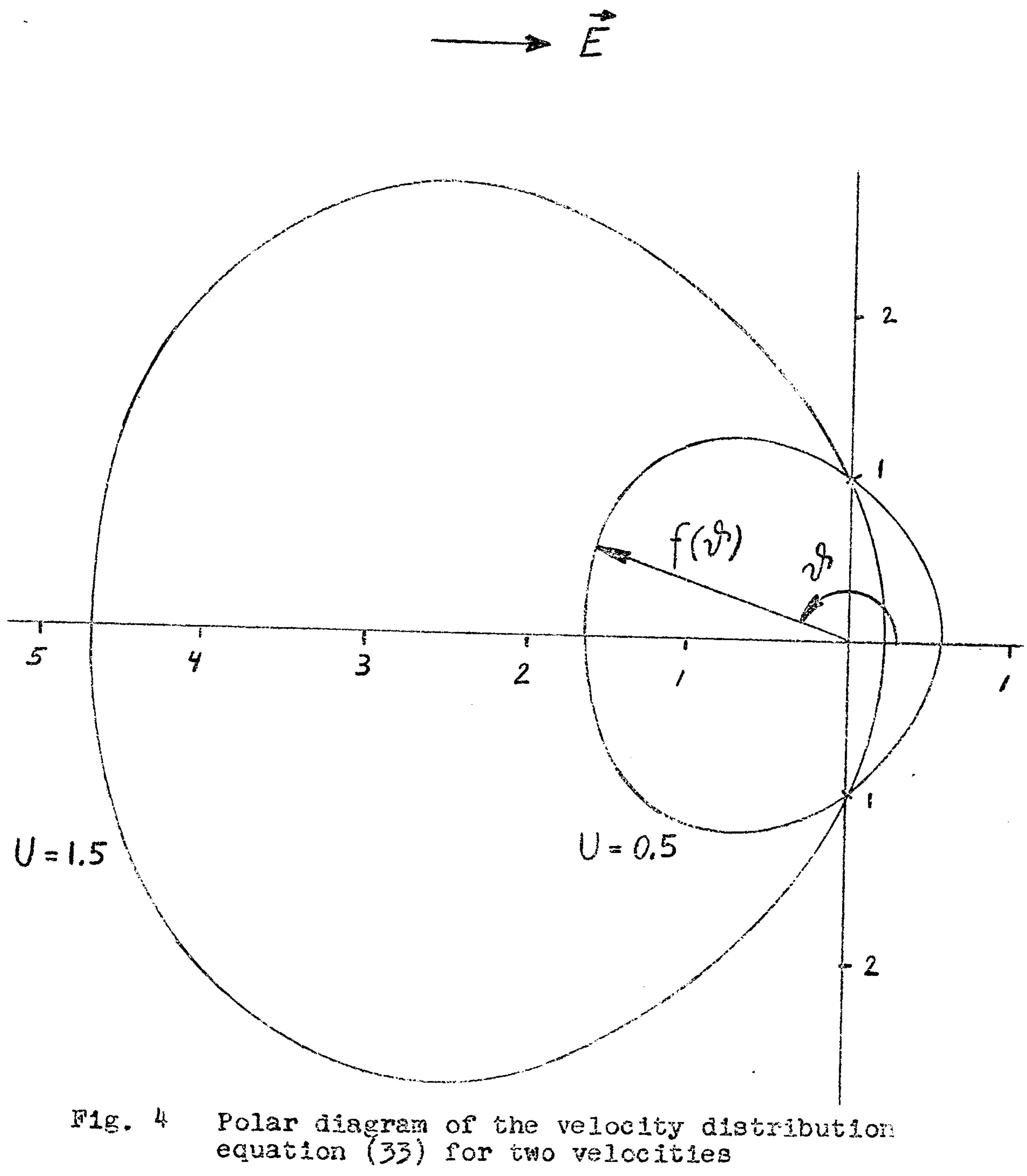

Fig. 4 Polar diagram of the velocity distribution equation (33) for two velocities

$$U = \frac{mn_o Q}{2eE} |\vec{v}|^2$$

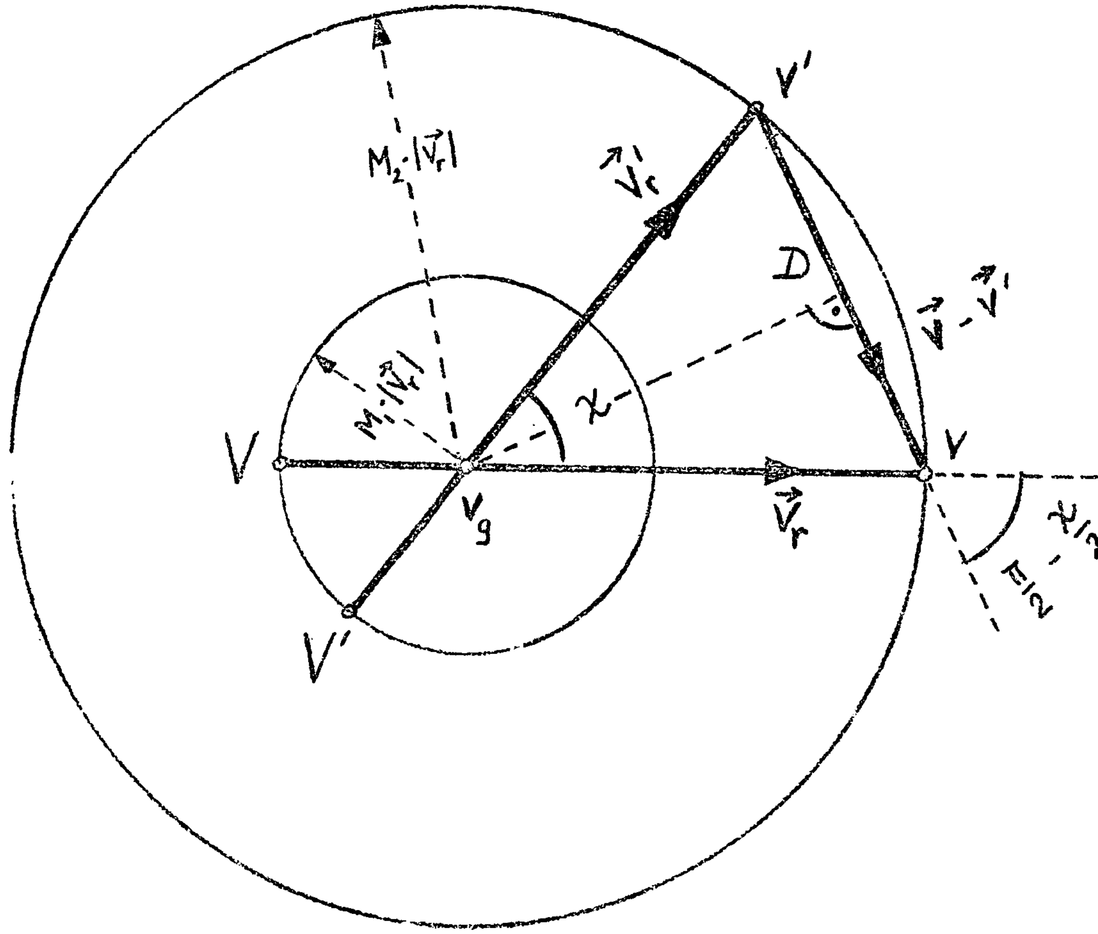

Fig. 5: Plane of relative motion $\perp \vec{p}$

The points labeled v, V, v, V', v_g represent the end points of the corresponding vectors $\vec{v}$, $\vec{V}$, $\vec{v}$, $\vec{V}'$, $\vec{v}_g$ in velocity space.

χ = scattering angle in the c.g. system

$\vec{v}_r = \vec{v} - \vec{V}$ relative velocity before collision

$\vec{v}_r' = \vec{v}' - \vec{V}'$ relative velocity after collision

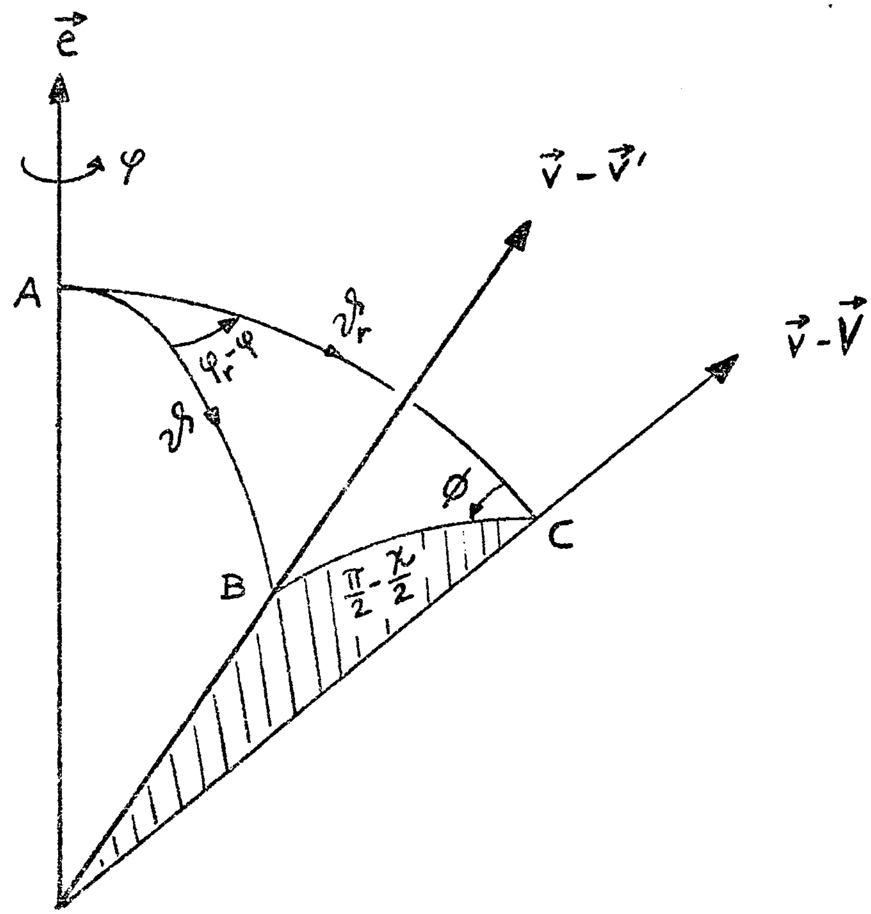

Fig. 6: Orientation of the plane of relative motion (shaded) with respect to an axis $\vec{e}$ fixed in space.

The velocity vectors $\vec{v}-\vec{v}'$ and $\vec{v}-\vec{V}$ are cut by a unit sphere and coincide with those shown in Fig. 5.

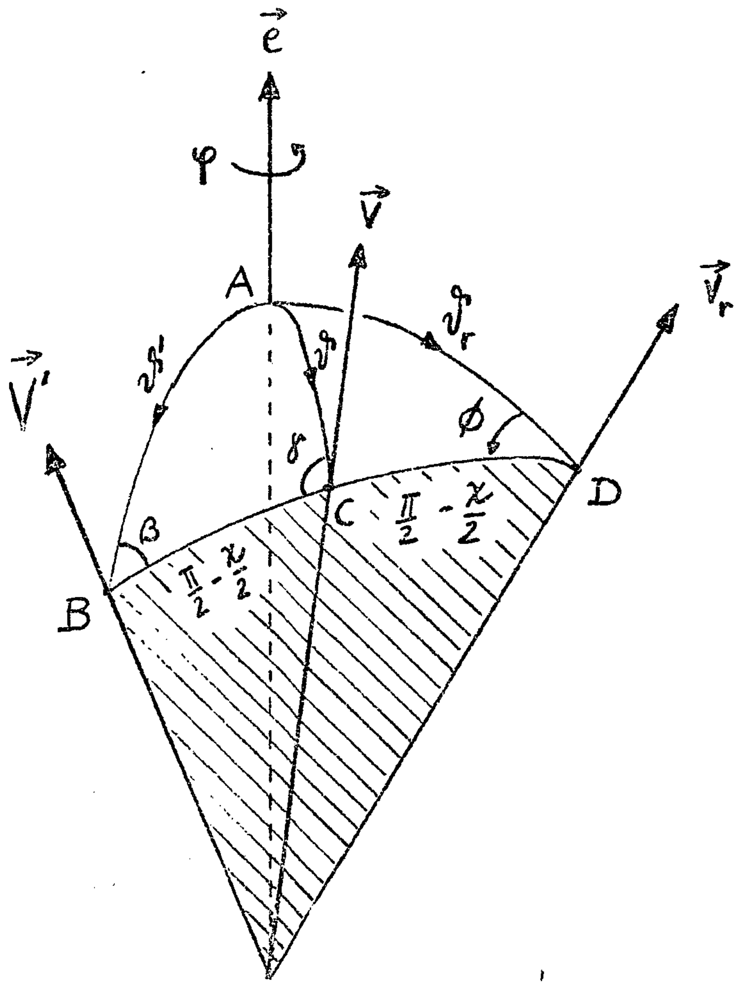

Fig. 7: Orientation of the plane of relative motion (shaded) with respect to an axis fixed in space, subject to the condition that one of the particles is at rest "after" collision $\vec{v}' = 0$.

The velocities $\vec{v}$, $\vec{V}'$, $\vec{v}_r$ are cut by a unit sphere and coincide with those shown in Fig. 3.